Stabilizing effect of periodic or eventually periodic constant pulses on chaotic dynamics

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It is shown that periodic or eventually periodic constant pulses applied to a chaotic dynamic may stabilize the system's trajectory at a periodic orbit. Methods to find all points where stabilization is possible and to calculate the corresponding constant pulses are given. Both discrete and continuous dynamics are considered. [S1063-651X(98)11006-1]

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In the last decades, several methods have been proposed to control chaotic dynamics, i.e., to stabilize the dynamics at a fixed level or at a periodic orbit. In their pioneering work, Ott, Grebogi, and York [1] showed that very small changes of a parameter, when appropriately performed, can effectively control a chaotic dynamic. The other methods to master chaos published since that time used proportional feedback, delay feedback [2,3], small periodic perturbations of a parameter [4], or regular pulses on a variable [5,6]. Interaction of a dynamic with one of its subsystems may synchronize chaotic dynamics [7]. Methods to control chaos have been applied to laboratory physics [8–11], chemistry [12], and experimental cardiology [13]. Good surveys of the topic are [14,15]. A book was published on the control of chaos, including reprints of 20 selected papers [16].

In a recent paper [17] we have examined the effects of periodic proportional pulses, $X_i \rightarrow kX_i$, performed on the trajectory, X_i, of a chaotic dynamic. This maneuver, suggested by Güémez and Matias [5,6], may stabilize the trajectory at a fixed point, or more generally, at a periodic orbit. We showed [17] for a given period p where stabilization is possible and, when this is possible, how to calculate the corresponding pulse k.

In the spirit of Matias and Güémez's method, we studied, in the present work, the possibility to stabilize chaos by subtracting constant amounts K from (or adding constant amounts K to) the variables of the dynamic, in a periodic manner. These interventions will be called ''periodic constant pulses.'' Both discrete and continuous dynamics are considered. Given an arbitrary integer p, we found all the points M in the phase space where constant pulses can stabilize the orbit at a periodic orbit of period p crossing M and showed how to calculate the corresponding constant K. The presentation of this paper is very similar to the previous one [17].

Consider first a d-dimensional discrete dynamic

$$X(n+1)=F(X(n)), \tag{1}$$

where $X=(x_1,x_2,\ldots,x_d)$, and F is a map of a domain D of $\mathbb{R}^d$ into itself. To perform the control, we subtract constant amounts $K=(k_1,k_2,\ldots,k_d)$ from X, once every p iterations. The constants $k_1,k_2,\ldots,k_d$ may be positive, negative, or zero, but we must be careful that the control will not move the orbit out of the domain of definition of F. Therefore the equation for the control is

$$X(i)\rightarrow X(i)-K \tag{2}$$

if $X(i)-K\in D$ and if i is a multiple of p.

In the case of a continuous dynamic

$$X'(t)=F(X(t)) \tag{3}$$

we assume that a Poincaré section which can be modeled in the form of Eq. (1), where n counts the returns of the orbit to the section, is available, and perform the pulses on the Poincar'e section once every p returns of the orbit to the section. At the beginning, the control is not periodic in time, but when the dynamic is stabilized at a periodic orbit, then the control becomes periodic in time. Details to calculate the control vector K corresponding to a given period p will be given below.

It is easy to show why periodic constant pulses may stabilize a discrete chaotic dynamic. Consider, for example, the logistic map $x(n+1)=f(x(n))=ax(n)[1-x(n)]$, with $a=3.9$. We recall that a fixed point x_s of a map f is locally stable if the derivative $f'(x_s)$ is smaller than 1 in absolute value. Let us pulse the dynamic with $p=2$ and $k=0.1$. Figures 1(a) and 1(b) show the maps $f^{(2)}(x)$ and $f^{(2)}(x)-k$. The map $f^{(2)}(x)$ has no stable fixed point while the map $f^{(2)}(x)-k$ has a stable fixed point [see point A in Fig. 1(b)]. Figures 1(c) and 1(d) show an orbit of the system $x(n+1)=f^{(2)}(x(n))$ and an orbit of the system $x(n+1)=f^{(2)}(x(n))-k$. So, the idea is very simple: we have just to localize the points x_s where $|f^{(2)\prime}(x_s)|<1$ and drive the curve $f^{(2)}(x)$ by $f^{(2)}(x)\rightarrow f^{(2)}(x)-k$ so that the curve $f^{(2)}(x)-k$ meets the diagonal at x_s. This gives the constant k.

For continuous dynamic having a Poincaré section that can be modeled, the control can be performed in the section as indicated above. We will give an example below.

Consider the discrete dynamic defined by Eq. (1), and let us perform the control as indicated by Eq. (2). Define

$$G(X)=F^{(p)}(X)-K \tag{4}$$

and $F^{(p)}$ is the p-times composition of the map F with itself. A fixed point of G is any solution X_s of the equation

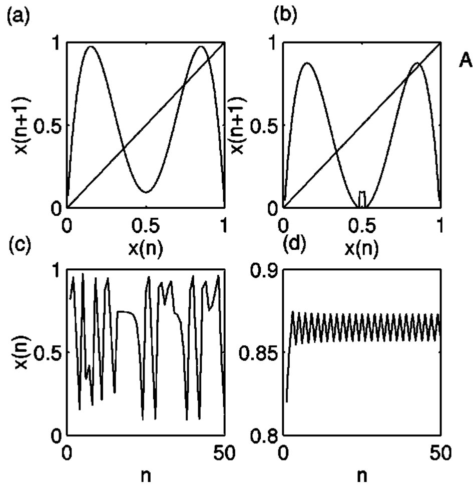

FIG. 1. (a) and (b) Maps $f^{(2)}(x)$ and $f^{(2)}(x)-0.1$, respectively, where $f(x)=3.9x(1-x)$. Point A in (b) is a fixed point of $f^{(2)}(x)-0.1$; (c) and (d) recurrent series $x(n)=f^{(2)}(x(n-1))$ and $x(n)=f^{(2)}(x(n-1))-0.1$, respectively.

$$G(X_s)=X_s \tag{5}$$

and this solution is locally stable if the Jacobian of G at X_s has all eigenvalues with modulus smaller than 1. Clearly, the Jacobian matrix of G is also the Jacobian matrix of $F^{(p)}$.

To stabilize the orbit at a period p, it is sufficient to find the points X_s such that the Jacobian of $F^{(p)}$ at X_s has all eigenvalues of modulus smaller than 1. Once the X_s have been identified, Eq. (5) gives the corresponding control vector K.

To give an example, consider the well-known Hénon map $x(n+1)=a-x(n)^2+by(n)$, $y(n+1)=x(n)$, with $a=1.4$ and $b=1/3$. Figure 2 shows the points (x_s,y_s) in the square $D=[-2,2]\times[-2,2]$ corresponding to $p=1$, 2, 3, and 4. To obtain Fig. 2, we have used a grid of 100×100 points in the square D, and, at each point, calculated the eigenvalues of $F^{(p)}$ ($p=1$, 2, 3, and 4) where F is the Hénon map. We plotted the point (x_s,y_s) if the corresponding eigenvalues

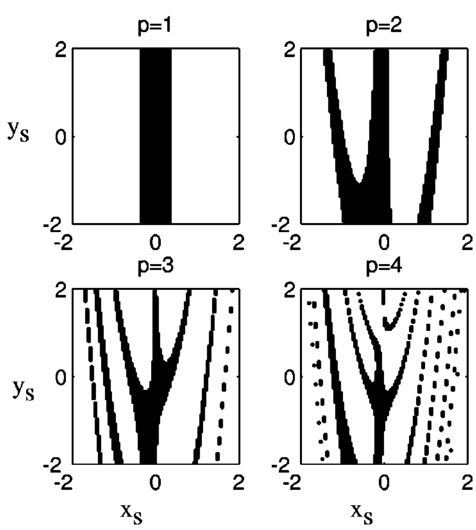

FIG. 2. Points (x_s,y_s) where one can stabilize the Hénon map at periods 1, 2, 3, and 4, by the constant pulses method.

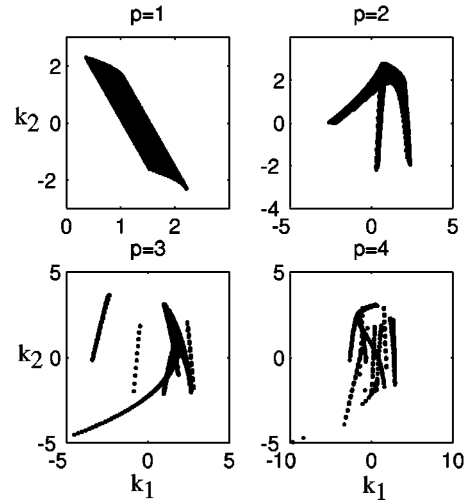

FIG. 3. Values of pulses $K=(k_1,k_2)$ to stabilize the Hénon map at periods 1, 2, 3, and 4.

have moduli lower than 1. The $K=(k_1,k_2)$ values corresponding to the points in Fig. 2, calculated by Eq. (5), are shown in Fig. 3.

Figures 2 and 3 show that, to stabilize the dynamic at a given period, we have some possibilities of choice. For example, for periods 1–4, one may stabilize the dynamic with $k_2=0$, with an appropriate choice of k_1.

The controlled orbit is not an orbit of the original map, unless $K=0$. Figure 3 shows that, for $p=1$ and 3, the point $K=(0,0)$ is not in the neighborhood of the plotted region, while this is the case for $p=2$ and 4. Therefore, with small values of K, our method can stabilize the Hénon orbit at periods 2 and 4 but not at periods 1 and 3. With small K values, the controlled orbit and the uncontrolled orbit are almost the same, the controlled orbit is locally stable while the uncontrolled orbit is unstable. Figure 4 shows two examples of $x(n)$: a series controlled to period 2 and a series controlled to period 4. When we performed the control (see

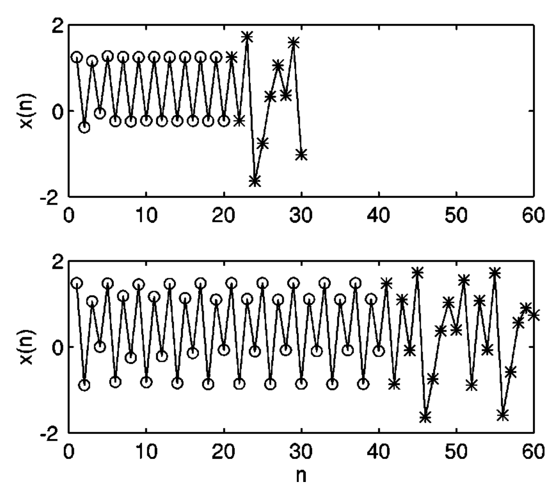

FIG. 4. Series $x(n)$ of the Hénon map, stabilized at periods 2 and 4 by small constant pulses. Under control (see the circle points), the series was quickly set to the desired periods. When control was dropped (see the starred points), the orbits remained at almost the same periodic orbit, but only for one more period, then became chaotic.

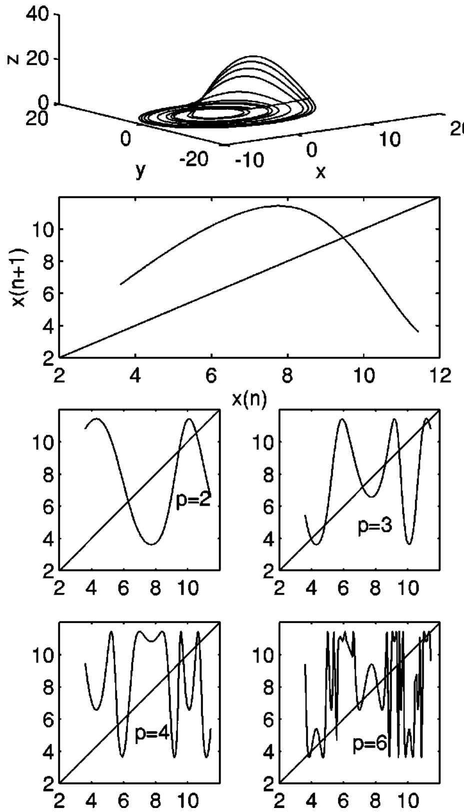

FIG. 5. Top: A Rössler orbit, with the Poincaré section corresponding to local maxima of the x component. The section was shown in the recurrent form $x(n)=f(x(n-1))$. Recurrent maps $f^{(p)}$ for $p=2$, 3, 4, and 6.

the circle points in Fig. 4), the orbit was stabilized to the desired period. When we dropped the control (see the starred points in Fig. 4), the trajectory remained almost at the same periodic orbit, but for just one more period, then became chaotic.

For continuous dynamics having a Poincaré section, it is sufficient to control the trajectory when it crosses the Poincaré section.

To give an example, let us consider the chaotic Rossler dynamic

$$x'(t)=-y(t)-z(t),$$

$$y'(t)=x(t)+0.2y(t), \quad (6)$$

$$z'(t)=0.2+z(t)[x(t)-5.7].$$

It is well known that this system has a good Poincaré section, defined by the points where the orbit cuts the Poincaré plane $y+z=0$, from the side $y+z<0$ to the side $y+z>0$, i.e., the points where $x(t)$ has a local maximal. We call these points the return points to the section. Let $(x(i),y(i),z(i))$ be the successive return points. A portion of the Rössler trajectory is shown in Fig. 5 (top), where the Poincaré section is the line transversal to the trajectory. Figure 5 shows also the depiction of $x(i+1)$ versus $x(i)$. This represents a quite smooth curve. We modeled the curve by the equation $x(n+1)=f(x(n))$, where f is a polynomial of degree 6. The polynomial was also drawn in Fig. 5, but the curve fitting was so good that one cannot distinguish the polynomial curve and the original points. Figure 5 (lower parts) shows the maps $f^{(p)}$ with $p=2$, 3, 4, and 6. For $p=2$ and 4, the graph of $f^{(p)}$ cuts the diagonal with very steep slopes, while for $p=3$ and $p=6$, the largest intersection of $f^{(p)}$ with the diagonal is very close to a local maximum. Therefore, for $p=3$ and $p=6$ (but not for $p=2$ and 4), we can, by a small pulse, obtain a curve $f^{(p)}-k$ which cuts the diagonal at a point with small derivative. We have shown the case $p=6$, but not the case $p=5$, because for $p=5$, one cannot stabilize the orbit with small values of k.

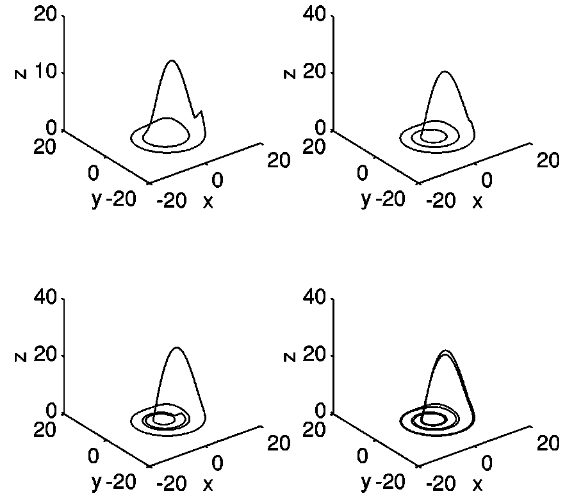

FIG. 6. Examples of Rössler's orbits controlled to periods 2, 3, 4, and 6, using the constant pulses procedure.

Consider only ''low'' periods, say, $p=1$ to 6 (for the moment, the period was counted as the number of the orbit's returns to the Poincaré section). To stabilize the dynamic, we may proceed as follows. When the orbit returns to a point (x_s,y_s,z_s) in the Poincaré section, we examine the recurrent map (Fig. 5) to see whether $|f^{(p)}(x_s)|<1$ for $p=1$ to 6. If it is so for a given p, we calculate the corresponding constant k by Eq. (2). At every pth subsequent return of the orbit to the Poincaré section, we perform the pulse with the constant k. Figure 6 shows the stabilized orbits, with $p=2$, 3, 4, and 6. We showed two examples where the k values are small ($p=3$ and $p=6$) and two examples where the k values are large ($p=2$ and $p=4$). These results were obtained by inspecting Fig. 5.

Notice the orbit issued from any point will reach the Poincaré section at a certain time. Therefore the present method examines the possibility to stabilize the orbit through any point in the phase space where the map is defined.

To conclude, this study investigates the possibility to stabilize a chaotic trajectory by applying periodic constant pulses to the orbit. The interest of the method is its simplicity. Comparing to several other methods to control chaos, the present maneuver is probably the simplest one.

There is a large possibility to control chaotic dynamic by a constant periodic pulse. If one wishes to maintain the original dynamic but stabilize it to a periodic orbit, one may choose the smallest K value. In the case of a chaotic population dynamic, where constant pulse represents the action of ''harvesting,'' one may wish to stabilize the dynamic at a periodic (and hence predictable) trajectory, with the most beneficial harvest. In that case, one may control with the largest K values.

Our examples suggest that periodic constant pulses applied to a chaotic dynamic have a potential ability to stabilize the orbit. It is conceivable that, without any knowledge of the dynamic and without any measurement, by just blindly performing regular constant pulses, one may stabilize a chaotic dynamic. The stabilizing effect of constant pulses on chaotic dynamics might be of interest in practical problems where the dynamic equations are unknown. It may help to understand the connection between periodic and chaotic phenomena in nature.

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